Remarks on the undecidability of the quantum halting problem

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Abstract

The halting problem is a decision problem first posed and proved by Alan Turing in 1936. With the recent surge of interest in quantum computation, one is led to ask if the problem can also be considered for a quantum computer. It is reported that the halting problem may not be solved consistently in both the Schrödinger and Heisenberg pictures of quantum dynamics. The assumption of the existence of the quantum halting machine leads to a contradiction when a vector representing an observable is the system that is to be unitarily evolved in both pictures.

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The classical halting problem asks whether it is possible to determine if the computation would halt for a given input [1]. With the construction of universal quantum Turing machine, Deutsch proposed [2] a quantum version of the halting problem. In case of quantum computation, we wish to determine if it is possible to decide if a halt qubit, initially set to 0, would be changed to 1 for completion of an arbitrary quantum algorithm applied to a quantum system which remains 0 otherwise. Previous discussions on the quantum halting problem have mainly focused on entanglement between a halt qubit and a system [3, 4, 5, 6]. In this paper, the halting scheme is approached differently and two pictures of quantum dynamics (i.e., the Schrödinger and Heisenberg pictures) are used. Schrödinger's wave mechanics and Heisenberg's matrix mechanics were formulated in the early twentieth century and have been considered to be equivalent. Therefore, in order to consider a halting scheme for a quantum system, one needs to examine whether the procedure is consistent in both the Schrödinger and Heisenberg pictures. An example in quantum dynamics is reported that shows this consistency may not be achieved.

In order to discuss the halting problem, let us first consider notations to be used. In particular a similar notation used in ([7], p243) will be introduced such that it is convenient in both the Schrödinger and Heisenberg pictures. A qubit in a density matrix form is written as $|\psi\rangle\langle\psi| = \frac{1}{2}(\mathbf{1} + \hat{\mathbf{v}} \cdot \vec{\sigma})$ where $\hat{\mathbf{v}} = (\mathbf{v}_x, \mathbf{v}_y, \mathbf{v}_z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ with $\sigma_x = |0\rangle\langle 1| + |1\rangle\langle 0|$, $\sigma_y = -i|0\rangle\langle 1| + i|1\rangle\langle 0|$, and $\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|$. Therefore a qubit, $|\psi\rangle\langle\psi|$, can be represented as a unit vector $\hat{\mathbf{v}} = (\mathbf{v}_x, \mathbf{v}_y, \mathbf{v}_z)$ pointing in (θ, ϕ) of a sphere with $0 \le \theta \le \pi, 0 \le \phi \le 2\pi$. A unitary transformation of a qubit in the vector notation $\hat{\mathbf{v}}$ can be obtained by applying U to σ_i for the corresponding ith component of the vector $\hat{\mathbf{v}}$, i.e., $\mathbf{v}_i$, where $i = x, y, z$. The transformation of $\hat{\mathbf{v}}$ under the unitary operation U will be written as $\hat{\mathbf{v}}' = U\hat{\mathbf{v}}U^\dagger$, implying the unitary transformation is applied to the corresponding σ_i. For example, let us consider the case when U is a rotation about y-axis by α in the Bloch sphere, i.e., $U = U_y \equiv \cos\frac{\alpha}{2}|0\rangle\langle 0| - \sin\frac{\alpha}{2}|0\rangle\langle 1| + \sin\frac{\alpha}{2}|1\rangle\langle 0| + \cos\frac{\alpha}{2}|1\rangle\langle 1|$. If we assume the initial state to be $\hat{\mathbf{v}} = (0, 0, 1)$, then U_y transforms it into $\hat{\mathbf{v}}' \equiv U_y\hat{\mathbf{v}}U_y^\dagger = (\sin\alpha, 0, \cos\alpha)$. In quantum theory, there is another important variable called an observable. For a single qubit, an observable can also be written as a unit vector, $\hat{\mathbf{e}} = (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z) = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)$, pointing (ϑ, φ) direction in a sphere. Therefore if one is to make a measurement in (ϑ, φ) direction, the observable would be $\hat{\mathbf{e}} \cdot \vec{\sigma}$. In the Heisenberg picture of quantum theory, it is the basis vector $\hat{\mathbf{e}}$ that is transformed. Using a similar transformation rule as in $\hat{\mathbf{v}}$, a unitary transformation of the observable in the basis vector notation can be obtained by applying $U^\dagger$ to the σ_j by $U^\dagger \sigma_j U$ for $\mathbf{e}_j$ which is to be represented as $\hat{\mathbf{e}}' = U^\dagger \hat{\mathbf{e}} U$. As an example, let us again consider the case when U is a rotation about y-axis by α, i.e. $U = U_y$. If vector $\hat{\mathbf{e}}$ is initially set to point in z-direction, i.e. $\hat{\mathbf{e}} = (0, 0, 1)$, then the transformation is as follows $\hat{\mathbf{e}}' \equiv U_y^\dagger \hat{\mathbf{e}} U_y = (-\sin\alpha, 0, \cos\alpha)$. As shown in Fig. 1, the directions of transformation for two vectors are different for Schrödinger and Heisenberg pictures. Therefore the expectation value $\hat{\mathbf{e}}' \cdot \hat{\mathbf{v}}$ in the Heisenberg picture remains the same as in the case with the Schrödinger picture, i.e., $\mathbf{e} \cdot \hat{\mathbf{v}}'$. For the remainder of this paper, the two vectors $\hat{\mathbf{v}}$ and $\hat{\mathbf{e}}$ will be treated on an equal footing. The only specialty about $\hat{\mathbf{e}}$ is that it serves as a coordinate or a basis vector such that when a measurement is made on the vector $\hat{\mathbf{v}}$, the expectation value is with respect to $\hat{\mathbf{e}}$.

With a quantum system and a halt qubit, Deutsch introduced [2] a quantum version of the halting problem wherein the completion of every valid quantum algorithm through a unitary process applied to the quantum system is accompanied by the change in a halt qubit to 1 that remains 0 otherwise. Let us assume such a halting machine exists. With the introduced notations, one particular case of the halting machine will be considered, that is, when the halting machine consists of a vector $\hat{\mathbf{v}}_\mathbf{s} \equiv (0, 0, 1)$ and a halt qubit $\hat{\mathbf{v}}_\mathbf{h} \equiv (0, 0, 1)$. An ancilla state is not included because it will not be needed for our discussion. The time evolution of the halting machine is defined through a unitary process, and the machine halts when the vector $\hat{\mathbf{v}}_\mathbf{s}$ is successfully rotated by α about y-axis. This time evolution of the halting machine can be achieved with the vector $\hat{\mathbf{v}}_\mathbf{s}$ evolving as follows

$$\hat{\mathbf{v}}_\mathbf{s} \rightarrow U_y \hat{\mathbf{v}}_\mathbf{s} U_y^\dagger \tag{1}$$

and the halt qubit $\hat{\mathbf{v}}_\mathbf{h}$ is transformed into $-\hat{\mathbf{v}}_\mathbf{h}$ with a unitary operation σ_x. In the following, we will show that this halting machine cannot achieve this task consistently in both pictures of quantum dynamics.

Suppose a closed system is consisted of a quantum state, represented by the vector $\hat{\mathbf{v}}_\mathbf{s} = (0, 0, 1)$, a basis vector representing an observable $\hat{\mathbf{e}}_\mathbf{s} = (0, 0, 1)$, and a halt qubit $\hat{\mathbf{v}}_\mathbf{h} = (0, 0, 1)$ along with the observable for the halt qubit defined as $\hat{\mathbf{e}}_\mathbf{h} \equiv (0, 0, 1)$. The vector $\hat{\mathbf{v}}_\mathbf{s}$ is to be transformed by α about y-axis, i.e. with U_y, and also σ_x is applied on the halt qubit such that $\hat{\mathbf{v}}_\mathbf{h} \rightarrow -\hat{\mathbf{v}}_\mathbf{h}$. If the evolved vector state is to be measured, the expectation value would be $\hat{\mathbf{e}}_\mathbf{s} \cdot (U_y \hat{\mathbf{v}}_\mathbf{s} U_y^\dagger)$. Next, let us consider the same procedure in the Heisenberg picture.

In the Schrödinger picture discussed above, the unitary evolution was performed on $\hat{\mathbf{v}}_{\mathbf{s}}$. Therefore, in the Heisenberg picture, $U_y^\dagger$ transforms the basis vector $\hat{\mathbf{e}}_{\mathbf{s}}$ into $U_y^\dagger \hat{\mathbf{e}}_{\mathbf{s}} U_y$ and the observable for the halt qubit, i.e., $\hat{\mathbf{e}}_{\mathbf{h}}$, is transformed into $-\hat{\mathbf{e}}_{\mathbf{h}}$. It yields the expectation value of $(U_y^\dagger \hat{\mathbf{e}}_{\mathbf{s}} U_y) \cdot \hat{\mathbf{v}}_{\mathbf{s}}$ which is equal to the expectation value in the Schrödinger picture, $\hat{\mathbf{e}}_{\mathbf{s}} \cdot (U_y \hat{\mathbf{v}}_{\mathbf{s}} U_y^\dagger)$.

Let us now consider the halting machine in (1) with one particular input. That is, when the input to be transformed is the basis vector, i.e., $\hat{\mathbf{v}}_{\mathbf{s}} = \hat{\mathbf{e}}_{\mathbf{s}}$. Note that the vectors $\hat{\mathbf{v}}$ and $\hat{\mathbf{e}}$ are being treated on a equal footing. In the Schrödinger picture, the evolution is as follows, $\hat{\mathbf{e}}_{\mathbf{s}} \to U_y \hat{\mathbf{e}}_{\mathbf{s}} U_y^\dagger \equiv \hat{\mathbf{e}}''_{\mathbf{s}} = (\sin\alpha, 0, \cos\alpha)$, while the halt qubit is transformed as $\hat{\mathbf{v}}_{\mathbf{h}} \to -\hat{\mathbf{v}}_{\mathbf{h}}$. The same procedure is now considered in the Heisenberg picture. In this case, the basis vector is transformed as $\hat{\mathbf{e}}_{\mathbf{s}} \to U_y^\dagger \hat{\mathbf{e}}_{\mathbf{s}} U_y \equiv \hat{\mathbf{e}}'''_{\mathbf{s}} = (-\sin\alpha, 0, \cos\alpha)$ and $\hat{\mathbf{e}}_{\mathbf{h}} \to -\hat{\mathbf{e}}_{\mathbf{h}}$. As can be seen in Fig. 2, it is noted that $\hat{\mathbf{e}}''_{\mathbf{s}} \neq \hat{\mathbf{e}}'''_{\mathbf{s}}$ unless $\alpha = k\pi$ where $k = 0, 1, 2....$ For the example studied in the previous paragraph, the vector $\hat{\mathbf{v}}_{\mathbf{s}}$ has evolved, with respect to $\hat{\mathbf{e}}_{\mathbf{s}}$, into the same output in both Schrödinger and Heisenberg pictures. That is, the relative angle between the two vectors $\hat{\mathbf{v}}_{\mathbf{s}}$ and $\hat{\mathbf{e}}_{\mathbf{s}}$ remained the same in both pictures. Similarly, with respect to $\hat{\mathbf{e}}_{\mathbf{h}}$, the halt qubit, $\hat{\mathbf{v}}_{\mathbf{h}}$, halted in both pictures. However, in the case with $\hat{\mathbf{e}}_{\mathbf{s}}$ as an input that was just considered, while the halt qubit $\hat{\mathbf{v}}_{\mathbf{h}}$ halted on both occasions with respect to $\hat{\mathbf{e}}_{\mathbf{h}}$, the vector that is being evolved, i.e., $\hat{\mathbf{e}}_{\mathbf{s}}$, turned out as two generally different outputs in two pictures. This contradicts the assumption about the halting machine in (1) because the machine should yield an output that is a rotation of the input by α about y-axis and is unique. Therefore, it has been shown that the assumption on the existence of the quantum halting machine leads into a contradiction when one considers the input of the basis vector, which is transformed into two generally different outputs in the Schrödinger and Heisenberg pictures.

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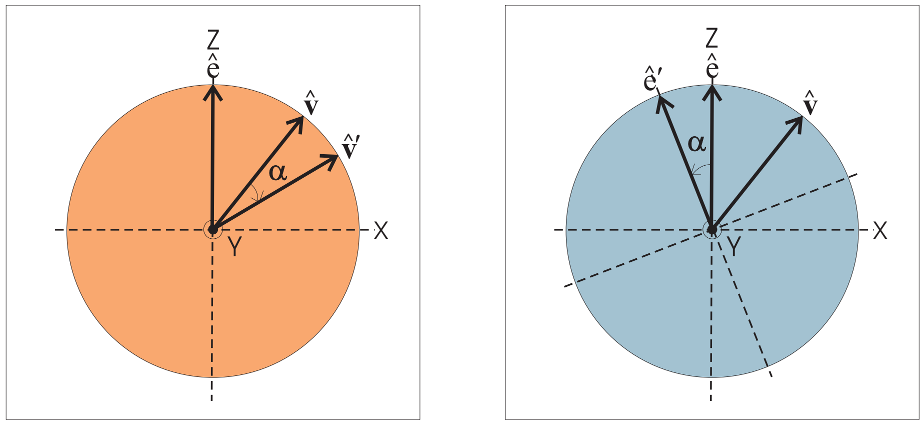
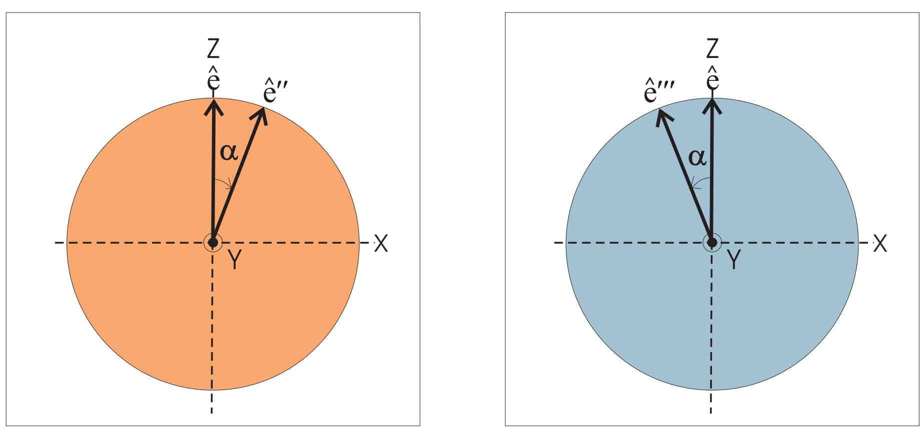


(a) Schrödinger picture (b) Heisenberg picture

FIG. 1: For the Schrödinger picture (a), the vector $\hat{\mathbf{v}}$ evolves while the basis vector $\hat{\mathbf{e}}$ is intact. In the Heisenberg picture (b), the basis vector $\hat{\mathbf{e}}$ is rotated into the opposite direction by the same amount while the vector $\hat{\mathbf{v}}$ remains, thereby keeping the angle between the two vectors, therefore the expectation values, the same in both pictures. With respect to $\hat{\mathbf{e}}$, the vector $\hat{\mathbf{v}}$ evolves by the same amount in both pictures.



(a) Schrödinger picture (b) Heisenberg picture

FIG. 2: Unitary evolution of $\hat{\mathbf{e}}$ is considered in both pictures. In the Schrödinger picture (a), $\hat{\mathbf{e}}$ is rotated clockwise by α while it is rotated counterclockwise by the same amount in the Heisenberg picture (b). The two outcomes are different unless $\alpha = k\pi$ where $k = 0, 1, 2, ...$ This leads into a contradiction since the halt qubit would be switched to 1 in both pictures for two generally different outcomes.